MAT102 Intro to Math Proofs

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Overview

Definition 3.1.1

A **mathematical statement** (or **proposition**) is a sentence that can be either **true** or **false**(in a given context).

Statement Examples

Identify whether the following examples are statements or not. If it is, is it true or false?

1. "The cube root of 8 is 2."
2. "$\mathbb{Q} \subseteq \mathbb{R}$"
3. "$11+11=110$"
4. "$x^2 > 0$"

Quantifiers

Let's turn "$x^2 > 0$" into a statement:

- "**For any** real number x, we have $x^2 > 0$."
- "$x^2 > 0$ **for some** real number x."

Can you identify quantifiers in the following examples?

1. $\forall x (x \geq 0)$ or "the square of any number is not negative."
2. $\forall x \forall y (x + y = y + x)$ i.e., the commutative law of addition.
3. $\exists x (x \text{ is a professor} \wedge x \text{ is an Asian})$ i.e. "some professor is Asian"

Connectives

The logical connectives of sentential logic are:

(a) Negation ("not"), denoted $\neg$.

(b) Conjunction ("and"), denoted $\wedge$.

(c) Disjunction ("or"), denoted $\vee$.

(d) Conditional ("if-then" or "implication"), denoted $\implies$.

(e) Biconditional ("if and only if" or "double implication"), denoted $\iff$.

The order of precedence of the logical connectives is:

1. Negation $\neg$
2. Conjunction $\wedge$
3. Disjunction $\vee$
4. Implication $\implies$
5. Double implication $\iff$

As usual, parentheses override the other precedence rules.

Can you find out what the following expression is equivalent to?

$$P \vee Q \wedge \neg R \implies S$$

Examples

1. "There are infinitely many primes and 2 divides 5."
2. "For any two natural numbers m and n, we have $n|m$ or $m|n$."
3. "Not every real number is rational."

An example using logic symbols

Suppose that
C = "The cheesesteak is good."
F = "The french fries are greasy."
W = "The wings are spicy."
Translate the following logical statements into words (with no logical symbols):
(a) $(\neg C \wedge F) \implies W$
(b) $\neg(C \vee W)$
(c) $\neg(\neg W \wedge C)$
(d) $\neg(\neg F)$.

More examples

- Suppose that a set B represents a group of parents, set A represents the group of their children. Denote by $P(x, y)$ the phrase "x is y's child." Do the two statements mean the same thing?

$$(\forall x \in A)(\exists y \in B)P(x, y)$$

$$(\exists y \in B)(\forall x \in A)P(x, y)$$

- Let $E(x)$ denote the phrase "x is even", and $O(x)$ denote "x is odd."

$$(\forall x \in \mathbb{Z})(E(x) \vee O(x)).$$

Truth Table

Assume that P and Q are two statements. Let R be the compound statement $P \vee Q$. We can find the **truth value** of R if we know the truth values of P and Q.

P	Q	$P \vee Q$

More truth tables

Can you fill up the truth tables?

P	$\neg P$

More truth tables

Can you fill up the truth tables?

P	Q	$P \wedge Q$

More truth tables

Can you fill up the truth tables?

P	Q	$P \iff Q$

Implications

Let P be the phrase "it is snowing", Q be the phrase "the temperature is less than or equal to $0^\circ C$."
Let

$R =$ If it is snowing, then the temperature is less than or equal to $0^\circ C$.

The structure is $P \implies Q$. Can you complete the truth table?

P	Q	$R = P \implies Q$

Consider two more examples:
1. If the tenant severely damages the property, the landlord has the right to terminate the lease.
2. If money grows on trees, then cats have five legs.

Consider the following statement R :

$$R = [P \wedge (\neg Q)] \implies [(\neg P) \vee Q].$$

The truth value of R will depend on the truth values of P and Q.

P	Q	$\neg Q$	$P \wedge (\neg Q)$	$\neg P$	$(\neg P) \vee Q$	R
T	T					
T	F					
F	T					
F	F					

Construct the truth table of

$$S = Q \vee [(\neg P) \iff (\neg Q)].$$

P	Q	$\neg P$	$\neg Q$	$(\neg P) \iff (\neg Q)$	S
T	T				
T	F				
F	T				
F	F				

Logically equivalent statements

Definition 3.4.1

Two statements are said to be **logically equivalent** if they always have the same truth value. In particular, two statements with the same truth table are logically equivalent.

Some equivalent statements

1. It's not the case that both the cheesecake is good and the french fries are greasy. It's equivalent to say that either the cheesecake is not good or the french fries aren't greasy.
2. It's not the case that either the cheesecake is good or the wings are spicy. It's equivalent to say that the cheesecake is not good and the wings aren't spicy.
3. Denote by P the phrase "being rich" and by Q the phrase "being happy." $\neg(P \implies Q)$ becomes "being rich doesn't imply being happy." The statement $P \wedge (\neg Q)$ translates to "one can be rich and unhappy."

4. You will receive A for a course if and only if your final grade is more than 80. It translates to if you receive A, then your final grade is more than 80 and if your final grade is more than 80, then you will receive A.
5. Let P denote "a person has a driver's license", and Q denote "being at least 16 years old". Consider the following statement: If a person has a driver's license, then he/she is at least 16 years old. It has the structure of $P \implies Q$. Consider another statement: If a person is not at least 16 years old, then he/she does not have a driver's license. It has the structure of $(\neg Q) \implies (\neg P)$.

Proposition 3.4.2

Let P and Q be two statements. Then the following pairs are logically equivalent.

- $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$.
- $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$.
- $\neg(P \implies Q)$ and $P \wedge (\neg Q)$.
- $P \iff Q$ and $(P \implies Q) \wedge (Q \implies P)$.
- $P \implies Q$ and $(\neg Q) \implies (\neg P)$.

Can you prove them?

Logically Equivalent statements

- The statements $\neg[\forall x P(x)]$ and $\exists x[\neg P(x)]$ are logically equivalent.
- The statements $\neg[\exists x P(x)]$ and $\forall x[\neg P(x)]$ are logically equivalent.

Let $n \in \mathbb{Z}$. If n^3 is odd, then n is odd.

A **tautology** is a statement that is always true. A **contradiction** is a statement that is always false.

Definition 3.5.1

The negation of a statement P is the statement $\neg P$ (not P).
For example, the negation of

$$P = \text{There is a set A, for which } A \in A.$$

is

$$\neg P = \text{There is } \textbf{no} \text{ set A, for which } A \in A.$$

Examples

Can you find the negation to the following statements?

- Let $U = \mathbb{R}$ be the set of all real numbers, and consider the following statement:

$$Q = \text{For all } a, b \in U, \text{if } a \cdot b = 0, \text{then } a = 0 \text{ or } b = 0.$$

- $S = (\forall a \in \mathbb{R})(\forall b \in \mathbb{R})[(a \cdot b = 0) \implies ((a = 0) \vee (b = 0))]$.
- $R = (\exists x \in (-3, 2) \cap \mathbb{Z})(x^2 < \frac{1}{2})$.
- $T = (\forall x \in \mathbb{R})[(x^2 + 1 < 0) \implies (15 < 5)]$.

A group of friends sits in a circle, each with a pile of wrapped candies. Some people have 20 or more pieces, others none, and the rest some number in between. The distribution is quite arbitrary except for the fact that everyone has been given an **even** number of pieces. A reserve supply is set aside.

The friends now follow these instructions: Give half your candy to the person on your left (and hence receive a supply of candy from the person on your right). Do this simultaneously .

Now recount your candy supply. If you now have an odd number of pieces, take an extra piece of candy from the reserve supply. This boosts your pile back up to an even number of pieces and everyone is ready to perform the maneuver again.

What happens to the distribution of candy among these friends if this maneuver is performed over and over again? Will people be forever taking extra pieces from the center, so everyone's amount of candy will grow without bound? Will one person end up with all the candy?